Teacher Support Materials 2009

Maths GCE

Paper Reference MFP4

Question 1

1 Let $\mathbf{P} = \begin{bmatrix} 1 & 4 & 2 \\ -1 & 2 & 6 \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} k & 1 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$, where k is a constant.

(a) Determine the product matrix **PQ**, giving its elements in terms of k where appropriate. *(3 marks)*

(b) Find the value of k for which **PQ** is singular. *(2 marks)*

Student Response

Leave blank

1

(a) $PQ = \begin{bmatrix} 1 & 4 & 2 \\ -1 & 2 & 6 \end{bmatrix} \begin{bmatrix} k & 1 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$

$= \begin{bmatrix} k+14 & -1 \\ 22-k & 7 \end{bmatrix}$ M1, A1, A0 2

(b) $7(k+14) - ((-1)22-k) = 0$

$7k + 98 + 22 - k = 0$ M1

$6k = -120$

$k = -20$ A1 2

4

Commentary

This candidate correctly realised that there are 2 rows in **P** (a 2 x 3 matrix) and 2 columns in **Q** (a 3 x 2 matrix) so the resulting product **PQ** will be a 2 x 2 matrix. However, the candidate was far from alone in making a slip in carrying out the calculation, suggesting that it would be wise to write down some working.

The solution to part (b) shows understanding that the determinant of a singular matrix is zero and the equation was correctly formed and solved. Although the final answer is incorrect, the error was in part (a), so a follow-through accuracy mark has been given.

Mark scheme

1(a)	$\begin{bmatrix} 1 & 4 & 2 \\ -1 & 2 & 6 \end{bmatrix}\begin{bmatrix} k & 1 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} k+14 & -1 \\ 22-k & 3 \end{bmatrix}$	M1 A1 A1	3	**PQ** a 2×2 matrix At least one element in C_1 correct All correct
(b)	$\text{Det}(\mathbf{PQ}) = 3k + 42 + 22 - k$ $= 2k + 64 = 0$ $k = -32$	M1 A1	 2	Det of a square matrix attempted and equated to zero ft in 2×2 case only (linear eqns.)
	Total		**5**	

Question 2

2 (a) Write down the 3×3 matrices which represent the transformations A and B, where:

(i) A is a reflection in the plane $y = x$; *(2 marks)*

(ii) B is a rotation about the z-axis through the angle θ, where $\theta = \dfrac{\pi}{2}$. *(1 mark)*

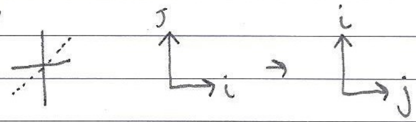
(b) (i) Find the matrix **R** which represents the composite transformation

'A followed by B' *(3 marks)*

(ii) Describe the single transformation represented by **R**. *(2 marks)*

Student response

2a i)



$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ ✓ B2

ii)

$$\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ ✓ B1

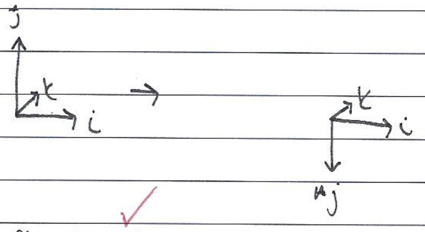
3

bi) $R = AB$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ B1 ✓ 1

ii)



reflection in the xz plane ✓ M1, A1 ✓ 2

(6)

Commentary

It was evident in this question that many candidates confused, for example, the $y = x$ plane with the y-x (or $z = 0$) plane. Although the diagram used here is very simple, it has enabled this candidate to confirm that a reflection in the plane $y = x$ swaps the x and y coordinates, leading directly to the correct matrix. Sensibly, the formula booklet was used to obtain the second matrix.
In part (b) the candidate has interpreted '**A** followed by **B**' as **AB** instead of **BA**. However, the matrix multiplication is correct so a follow-through mark has been given. In the final part, a diagram has again been used leading to a correct interpretation of the matrix. The follow-through marks would not have been given here had either **A** or **B** been incorrect.

2(a)(i)	$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	B2	2	
(ii)	$\mathbf{B} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	B1	1	
(b)(i)	$\mathbf{R} = \mathbf{BA} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	M1 A1 A1	3	Product correct way around Most correct; all correct ft ft
(ii)	Reflection in $x = 0$ (or y–z plane)	M1 A1	2	M for correct **R**
	<u>Note 1:</u> For $\mathbf{R} = \mathbf{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	(B1)		If all correct, ft their **A**, **B**
	Reflection in $y = 0$ (or x–z plane)	(M1) (A1)		Full ft, M for correct **R**
	<u>Note 2:</u> 90° rotation in –ve sense gives $\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	(B1)		**A** as before
	$\mathbf{R} = \mathbf{BA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Reflection in $y = 0$ (or x–z plane)	(M1) (A1) (A1) (M1) (A1)		Full ft (incl. Note 1 possibility – Reflection in $x = 0$ (or y–z plane))
	Total		**8**	

Question 3

3 The plane Π has equation $\mathbf{r} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$.

(a) Find an equation for Π in the form $\mathbf{r} \cdot \mathbf{n} = d$. *(4 marks)*

(b) Show that the line with equation $\mathbf{r} = \begin{bmatrix} 7 \\ 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 10 \\ 1 \\ 5 \end{bmatrix}$ does not intersect Π, and explain the geometrical significance of this result. *(4 marks)*

Student Response

3a) $n = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$ ✓

M1, A1

$n = \begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix}$ ✓ $\quad d = \begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ ✓ M1

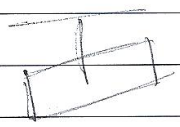
$d = 6 + 5 - 7$

$d = 4$ ✓ A1

$r \cdot \begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix} = 4$ ✓

4

b)



$\begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 1 \\ 5 \end{bmatrix} = 30 + 5 - 35$ ✓

$= 0$ ✓ M1, A1

$a \cdot b = |a||b| \cos\theta \qquad |a| \neq 0 \quad |b| \neq 0 \qquad \therefore \cos\theta = 0$

$\theta = \pi/2$

$\therefore$ plane's normal is perpendicular to the line.

B1

This means that the plane is parallel to the line.

A B0

3

7

Leave blank

Commentary

In part (a) the candidate shows understanding of the need to find a vector that is perpendicular to **both** the direction vectors of the plane, using the vector product of those two vectors. The calculation has been done without showing any working, but the correction made in the vector shows that it was, very sensibly, checked. Before carrying out the next step it would have good to offer some explanation, even writing down $\underline{\boldsymbol{r}}.\underline{\boldsymbol{n}} = d$, but again both method and arithmetic are correct.

In part (b), the obvious way of showing the line and plane do not intersect is to use the answer to part (a) and substitute for $\underline{\boldsymbol{r}}$ on the left hand side in order to show that the result is **not** equal to d. This was the method chosen by the majority of candidates. This candidate took a different approach, adopted by a substantial minority, of using the scalar product to show that the perpendicular to the plane and the direction vector of the line are perpendicular to each other. The plane and line are therefore parallel as shown in the candidate's sketch. However, only a couple of candidates, not including this one, realised that to show there was no **intersection**, they must also show that the line does not lie in the plane.

Mark Scheme

3(a)	$\mathbf{n} = (3\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \times (4\mathbf{i} - \mathbf{j} + \mathbf{k})$	M1		
	$= 3\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$	A1		cao
	$d = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) \bullet (\text{their } \mathbf{n}) = 4$	M1 A1	4	ft
(b)	$\begin{bmatrix} 7+10t \\ 1+t \\ 4+5t \end{bmatrix}$ subst^{d}. into their plane eqn.	M1		(In at least the LHS of it)
	$21 + 30t + 5 + 5t - 28 - 35t = 4$	dM1		Linear "eqn." in t created (LHS)
	Since $-2 \neq 4$, no intersection	A1		Explained or stated. N.B. can ft other d's (except -2) but if $\mathbf{n}$ is wrong also the t won't vanish, so no ft then
	Line parallel to plane	B1		May be independently asserted
	OR			
	$\begin{bmatrix} 3 \\ 5 \\ -7 \end{bmatrix} \bullet \begin{bmatrix} 10 \\ 1 \\ 5 \end{bmatrix} = 0$	(M1) (A1) (B1)		For showing line not in plane
	Line perp^{r}. to nml. $\Rightarrow$ line // to plane	(B1)		
	OR			
	$\begin{bmatrix} 7+10t \\ 1+t \\ 4+5t \end{bmatrix}$ equated to $\begin{bmatrix} 2+3\lambda+4\mu \\ 1+\lambda-\mu \\ 4+2\lambda+\mu \end{bmatrix}$	(M1)		Incl. starting to do something
	Eliminating λ, μ to get linear eqn. in t	(dM1)		
	Since $-2 \neq 4$, no intersection	(A1)		Explained or stated
	Line parallel to plane	(B1)	4	May be independently asserted
	Total		**8**	

Question 4

4 (a) Show that the system of equations

$$\begin{aligned} 3x - y + 3z &= 11 \\ 4x + y - 5z &= 17 \\ 5x - 4y + 14z &= 16 \end{aligned}$$

does not have a unique solution and is consistent.

(You are not required to find any solutions to this system of equations.) *(4 marks)*

(b) A transformation T of three-dimensional space maps points (x, y, z) onto image points (x', y', z') such that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x - y + 3z - 2 \\ 2x + 6y - 4z + 12 \\ 4x + 11y + 4z - 30 \end{bmatrix}$$

Find the coordinates of the invariant point of T. *(8 marks)*

Student Response

Question number | Leave blank

(4)
$$\begin{vmatrix} +3 & -{-1} & +3 \\ 4 & 1 & -5 \\ 5 & -4 & 14 \end{vmatrix}$$

$= 3(14 - 20) + 1(56 + 25) + 3(-16 - 5)$

$= -18 + 81 - 63 = 0$. ✓ M1

determinant $= 0$ $\therefore$ system of equations does not have a unique solution ✓ A1

$3x - y + 3z = 11$ ①

$4x + y - 5z = 17$ ②

$5x - 4y + 14z = 16$ ③

① + ②: $7x - 2z = 28$ ④ ✓ M1

4② + ③: $21x - 6z = 84$ ⑤ ✓

$④ = 3⑤$ $\therefore$ system of equations is consistent. ✓ A1

4

(b)
$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - y + 3z - 2 \\ 2x + 6y - 4z + 12 \\ 4x + 11y + 4z - 30 \end{bmatrix}$$

(4)

(6) continued.

$$\begin{bmatrix} x-y+3z-2 \\ 2x+6y-4z+12 \\ 4x+11y+4z-30 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$x-y+3z-2=x$ (1)

$2x+6y-4z+12=y$ (2) M1, A1

$4x+11y+4z-30=z$ (3)

$-y+3z=2$ (1)

$2x+5y-4z=-12$ (2)

$4x+11y+3z=30$ (3)

$2(2)-(3)$:

$-y-11z=-54$ (4) M1, A1

(1): $-y+3z=2$ (5)

(4) − (5) $-14z=-56 \Rightarrow z=-4$.

$\cancel{(4)\ -y+11(-4)=-54}$

$\cancel{y=44+54=98}$

(1) $-y+3(-4)=2$

$-y-12=2 \Rightarrow y=-14$ × M1, A0

Commentary

The approach taken by this candidate in part (a) was the most common and a brief but adequate explanation has been given as to the reason for the equations having no unique solution. There are signs that this candidate, knowing that the determinant should be zero has checked, and corrected, the original working. Sadly, some obtained a non-zero answer and, instead of checking, stated that the system **did** have a unique solution, in spite of being asked to show that it did not. In the second part of (a), by numbering the equations, and referring to those numbers, the candidate has made the steps clear to the examiner, thus ensuring full marks. This is very important, not only to gain marks, but also to enable candidates to check their own work easily.

Those candidates who tackled consistency before evaluating the determinant often knew that obtaining two identical equations meant that the system had an infinite number of solutions, demonstrating both of the conditions at once.

Part (b) had a hesitant start but perhaps the candidate re-read the question (as many evidently did not) and realised that the result of the transformation was given. All that was needed before solving was to equate the three rows of the matrix to x, y and z respectively. From this point on the solution was set out clearly and efficiently and it was a pity that a slip led to the loss of two accuracy marks. Candidates who are less organised than this one may find that using the augmented matrix method provides a helpful structure to their work.

4(a)	$3 \times [1] - [2] \Rightarrow 5x - 4y + 14z = 16$	M2 A1		Or eliminating (say) y twice to get two lots of $7x - 2z = 28$
	Giving no unique soln. *and* consistent	E1		
	For those who just show $\Delta = 0$ to conclude that there is no unique soln.	(M1) (A1)		and save the other M1 A1 for demonstrating consistency
	OR Solving e.g. in [1] & [2]: $\frac{x-4}{2} = \frac{y-1}{27} = \frac{z}{7} = \lambda$	(M1) (A1)		
	Subst^{n}. in [3] for x, y, z in terms of λ Showing LHS = RHS = 16	(M1) (A1)		$5(2\lambda + 4) - 4(1 + 27\lambda) + 14(7\lambda)$
	OR $\left(\begin{array}{ccc\|c} 3 & -1 & 3 & 11 \\ 4 & 1 & -5 & 17 \\ 5 & -4 & 14 & 16 \end{array}\right) \rightarrow \left(\begin{array}{ccc\|c} 3 & -1 & 3 & 1 \\ 1 & 2 & -8 & 6 \\ -1 & -2 & 8 & -6 \end{array}\right)$	(M1) (A1) (A1)		$R_2' = R_2 - R_1$ $R_3' = R_3 - 2R_1$
	$R_2' = -R_3' \Rightarrow$ no unique soln. and consistency	(E1)		
	OR Showing $\Delta = 0 \Rightarrow$ no unique soln.	(M1) (A1)		
	Attempt at each of $\Delta_x = \begin{vmatrix} 11 & -1 & 3 \\ 17 & 1 & -5 \\ 16 & -4 & 14 \end{vmatrix}$, $\Delta_y = \begin{vmatrix} 3 & 11 & 3 \\ 4 & 17 & -5 \\ 5 & 16 & 14 \end{vmatrix}$ and $\Delta_z = \begin{vmatrix} 3 & -1 & 11 \\ 4 & 1 & 17 \\ 5 & -4 & 16 \end{vmatrix}$	(M1)		
	Each shown = 0 and this $\Rightarrow$ consistency	(A1)	4	
(b)	Setting $x' = x,\ y' = y,\ z' = z$ $2 = -y + 3z$ $-12 = 2x + 5y - 4z$ $30 = 4x + 11y + 3z$	M1 A1		Or equivalent
	E.g. $\left.\begin{array}{l} 2 = 3z - y \\ 54 = 11z + y \end{array}\right\}$ by (3) – 2 × (2)	M1 A1		Reducing to 2×2 system; Correctly ft their system
	$z = 4,\ y = 10$ $x = -23$	M1 A1 M1 A1	8	Solving ; correctly Subst^{n}. back to find 3rd coord.
	OR Other methods for solving a 3×3 system will be constructed should they arise			
	Total		**12**	

Question 5

5 The points A, B, C and D have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ respectively, relative to the origin O, where

$$\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} \text{ and } \ \mathbf{d} = \begin{bmatrix} 5 \\ 5 \\ 11 \end{bmatrix}$$

(a) Using scalar triple products:

(i) show that $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are coplanar; *(2 marks)*

(ii) find the volume of the parallelepiped defined by AB, AC and AD. *(4 marks)*

(b) (i) Find the direction ratios of the line BD. *(2 marks)*

(ii) Deduce the direction cosines of the line BD. *(2 marks)*

Student Response

Question number

Leave blank

5. a. i. $\underline{a}.\underline{b}\times\underline{c} = \begin{vmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 1 & -1 & 5 \end{vmatrix} = 2(10+5) - 1(15-5) + 4(-3-2)$ ✓M1

$= 30 - 10 - 20 = 0$

$\therefore$ volume $= 0$ $\therefore$ coplanar ✓A1 — 2

ii. $AB = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ✓ $AC = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ ✓ $AD = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}$ ✓ M1, A1

Vol. $= |\underline{AB}.\underline{AC}\times\underline{AD}|$ ✓ $= \left\| \begin{matrix} 1 & 1 & 1 \\ -1 & -2 & 1 \\ 3 & 4 & 7 \end{matrix} \right\| = |1(-14-4) - 1(-7-3) + 1(-4+6)|$ M1

$= |-18 + 10 + 2| = |-6| = 6$ ✓ A1 — 4

b. i. BD gradient : $\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ ✓ M1 ratio $x : y : z$

$2 : 3 : 6$ ✓ A1 — 2

ii. $\cos\theta_x = \dfrac{2}{\sqrt{2^2+3^2+6^2}}$ ✓M1 $= \dfrac{2}{7}$ ✓

$\cos\theta_y = \dfrac{3}{7}$ ✓

$\cos\theta_z = \dfrac{6}{7}$ ✓ A1 — 2

(10)

Commentary

This is an excellent solution to the question. The candidate not only realised that the zero result to the triple product means that the vectors are coplanar, but also said so. In part (a)(ii), the triple product is correctly evaluated as -6 but, again correctly, the volume has been given as 6.
In part (b) this candidate knew what many did not; direction ratios are ratios and come directly from the direction vector. The equation of the line was not required. The final part was also correct. Full marks.

5(a)(i)	$\mathbf{a}\bullet\mathbf{b}\times\mathbf{c} = \begin{vmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 1 & -1 & 5 \end{vmatrix} = 0$	M1 A1	2	Legitimately shown to be zero
(ii)	$\overrightarrow{AB} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \overrightarrow{AC} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \ \overrightarrow{AD} = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$	M1 A1		At least two correct
	Attempt at $\overrightarrow{AB} \bullet \overrightarrow{AC} \times \overrightarrow{AD}$	M1		Any order (+/–), some Sc.Trip.Pr.
	$V = 6$	A1	4	cao and not –ve
(b)(i)	$\overrightarrow{BD} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$; i.e. $2 : 3 : 6$	M1 A1	2	
(ii)	$\sqrt{2^2 + 3^2 + 6^2} = 7$	M1		
	DCs are $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$	A1	2	ft
	Total		**10**	

Question 6

6 The plane transformation T is defined by

$$\mathrm{T} : \begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{M}\begin{bmatrix} x \\ y \end{bmatrix}$$

where $\mathbf{M} = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$.

(a) Evaluate $\det \mathbf{M}$ and state the significance of this answer in relation to T. *(2 marks)*

(b) Find the single eigenvalue of $\mathbf{M}$ and a corresponding eigenvector. Describe the geometrical significance of these answers in relation to T. *(5 marks)*

(c) Show that the image of the line $y = \frac{1}{2}x + k$ under T is $y' = \frac{1}{2}x' + k$. *(3 marks)*

(d) Given that T is a shear, give a full geometrical description of this transformation. *(2 marks)*

Student Response

1. a $|M| = \begin{vmatrix} -1 & 4 \\ -1 & 3 \end{vmatrix} = -3 + 4 = 1$ ✓ B1

The transformation T does not change the area of the shape it transforms. ✓ B1 2

b $\begin{vmatrix} -1-\lambda & 4 \\ -1 & 3-\lambda \end{vmatrix} = (\lambda+1)(\lambda-3) + 4$

$= \lambda^2 - 2\lambda + 1$

$= (\lambda - 1)(\lambda - 1)$

$\lambda = 1$ ✓✓ M1, A1

The ~~a~~ single eigen value is 1

$\left(\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & 2 & 0 \end{array}\right)$ $-x + 2y = 0$ ✓ M1

$x = 2y$

eigenvector $= \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ✓ A1

There is a line of invariant points ~~at~~ with the direction $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ✓ after a transformation T B1 5

Question number		Leave blank
c	$\begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ \frac{1}{2}x + k \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$	
	$\begin{pmatrix} -x + 2x + 4k \\ -x + \frac{3}{2}x + 3k \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$	
	$\begin{pmatrix} x + 4k \\ \frac{1}{2}x + 3k \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ ✓ M1, A1	
	$y' = \frac{1}{2}x' + k$ ∧ A0	2
d.	Two is shear parallel to the line $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ✓ - - - B1	
	∧ B0	1
		10

Commentary

A surprising number of candidates, whilst knowing how to evaluate a 2 x 2 determinant and writing down -3 + 4 went on to get -1 as the answer. No such problem here, and the required comment referring to area was then given.
The only lapse in part (b) is that the characteristic equation is not in fact shown as an equation, having no right-hand side. However, the solution for λ implies this and full marks were given. However, the omission of a right-hand side is a common practice that is not only incorrect but also frequently leads to errors. The rest of (b) is excellent, with the candidate correctly giving the geometrical significance of $\lambda = 1$ as a line of invariant points and not just an invariant line.
In part (c) most candidates, as here, correctly found the image of x and y by replacing y by ½$x + k$ and performing the matrix multiplication (although a substantial minority simply replaced x by x' and y and y'). However, this is a question where candidates are asked to **show** a result and it is therefore not enough just to see it themselves. It must be spelt out. In this case they needed to write out the two equations. The clearest way to continue is to rearrange the equation for x' into the form $x = x' - 4k$ and substitute into $y' = \frac{1}{2}x + 3k$ giving $y' = \frac{1}{2}(x' - 4k) + 3k$. Multiplying out the brackets gives the answer.
In the final part the candidate, by giving the eigenvector, has partly described the shear, although the equation of the invariant line is usually given. One mark has been lost by not giving an example of a mapping, such as (1,0) to (-1,-1).

6(a)	Det(**M**) = 1 $\Rightarrow$ Area invariant under T	B1 B1	2	2nd B1 ft ref. "area"
(b)	Char. Eqn. $\lambda^2 - 2\lambda + 1 = 0$ $\Rightarrow \lambda = 1$ (twice) Subst$^{\text{g}}$. their λ back to find an evec: $\alpha\begin{bmatrix}2\\1\end{bmatrix}$	M1 A1 M1 A1		 Any (non-zero) α
	(Since $\lambda = 1$) this represents a line of inv. pts.	B1	5	ft if $\lambda \neq 1$
(c)	$\begin{bmatrix}-1 & 4\\-1 & 3\end{bmatrix}\begin{bmatrix}x\\\frac{1}{2}x+k\end{bmatrix}=\begin{bmatrix}x+4k\\\frac{1}{2}x+3k\end{bmatrix}$	M1 A1		
	Verifying that $y' = \frac{1}{2}x' + k$	A1	3	Be convinced AG
(d)	Inv. line (or parallel to) $y = \frac{1}{2}x$	B1		
	Mapping (e.g.) (1, 0) to (–1, – 1) Give 0 + 0 if called any other kind of transformation	B1	2	Any pt. not on $y = \frac{1}{2}x$ and its image
	Total		**12**	

Question 7

7 The 2×2 matrix $\mathbf{M}$ has an eigenvalue 3, with corresponding eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and a second eigenvalue -3, with corresponding eigenvector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

The diagonalised form of $\mathbf{M}$ is $\mathbf{M} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}$.

(a) (i) Write down suitable matrices $\mathbf{D}$ and $\mathbf{U}$, and find $\mathbf{U}^{-1}$. *(4 marks)*

(ii) Hence determine the matrix $\mathbf{M}$. *(3 marks)*

(b) Given that n is a positive integer, use the result $\mathbf{M}^n = \mathbf{U}\,\mathbf{D}^n\,\mathbf{U}^{-1}$ to show that:

(i) when n is even, $\mathbf{M}^n = 3^n\,\mathbf{I}$;

(ii) when n is odd, $\mathbf{M}^n = 3^{n-1}\,\mathbf{M}$. *(6 marks)*

Student Response

7a) i. $D = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$ ✓ B1

$U = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$ ✓ B1

$U^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$ ✓ ✓ B1, B1 4

ii. $M = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} 3 & -3 \\ 6 & -12 \end{bmatrix}\begin{bmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$ ✓ M1, A1

$= \begin{bmatrix} 9 & -3 \\ 24 & -9 \end{bmatrix}$ ✓ A1 3

b) i. $M^n = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 3^n & 0 \\ 0 & (-3)^n \end{bmatrix}\begin{bmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$ ✓

$= \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 3^n & 0 \\ 0 & 3^n \end{bmatrix}\begin{bmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$ ✓ ($(-3)^n$ is positive when n is even) ✓

$= \begin{bmatrix} 3^n & 0 \\ 0 & 3^n \end{bmatrix}$ need $D = 3^n I$ M0

$= 3^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 3^n I.$ ✗ 0.

Question number

Leave blank

ii. $M^n = \begin{bmatrix}1 & 1\\ 2 & 4\end{bmatrix}\begin{bmatrix}3^n & 0\\ 0 & (-3)^n\end{bmatrix}\begin{bmatrix}2 & -\frac{1}{2}\\ -1 & \frac{1}{2}\end{bmatrix}$

$= \begin{bmatrix}1 & 1\\ 2 & 4\end{bmatrix}\begin{bmatrix}3^n & 0\\ 0 & -3^n\end{bmatrix}\begin{bmatrix}2 & -\frac{1}{2}\\ -1 & \frac{1}{2}\end{bmatrix}$ ✓ ($(-3)^n$ is negative when n is odd)

$= \begin{bmatrix}3^n & -3^n\\ 2\times 3^n & 4\times -3^n\end{bmatrix}\begin{bmatrix}2 & -\frac{1}{2}\\ -1 & \frac{1}{2}\end{bmatrix}$ ✓ M1, A1

$= \begin{bmatrix}3\times 3^{n-1} & -3\times 3^{n-1}\\ 6\times 3^{n-1} & -12\times 3^{n-1}\end{bmatrix}\begin{bmatrix}2 & -\frac{1}{2}\\ -1 & \frac{1}{2}\end{bmatrix}$

$= 3^{n-1}\begin{bmatrix}3 & -3\\ 6 & -12\end{bmatrix}\begin{bmatrix}2 & -\frac{1}{2}\\ -1 & \frac{1}{2}\end{bmatrix}$

$= 3^{n-1}\begin{bmatrix}9 & -3\\ 24 & -9\end{bmatrix}$ ✓ A1

$= 3^{n-1} M$ as required ✓

3

10

Commentary

This candidate's solution shows part (a) exactly as it should be, with the determinant of $\mathbf{U}$ correctly calculated to give the ½ for the inverse of $\mathbf{U}$.
Part (b) started well with brackets correctly used. Then the sophisticated, algebraic approach was chosen. Unfortunately although, in the 2nd line, $\mathbf{D}^n$ was written down there is no evidence that the candidate did not simply alter the order of the matrices and combine $\mathbf{UU}^{-1}$. This transgression of the rule for multiplying matrices was made explicitly by a number of candidates so the examiner could not give the benefit of the doubt in this example. Also, since the question asks for the result to be **shown**, marks cannot be given if steps are omitted. After the 2nd line, the candidate needed to show the substitution of $\mathbf{D}^n$ by writing $\mathbf{M}^n = \mathbf{U3^nIU}^{-1}$ followed by $= \mathbf{3^n UIU}^{-1}$ then $= \mathbf{3^n UU}^{-1}$ and finally the result.
In (b)(ii), the candidate chose the arithmetic approach, again with correct use of brackets. The solution was then completed accurately.

7(a)(i)	$\mathbf{D}=\begin{bmatrix}3 & 0\\ 0 & -3\end{bmatrix}$ $\quad \mathbf{U}=\begin{bmatrix}1 & 1\\ 2 & 4\end{bmatrix}$	B1 B1		**D**, **U** (alt. choices ok)
	$\mathbf{U}^{-1}=\frac{1}{2}\begin{bmatrix}4 & -1\\ -2 & 1\end{bmatrix}$	B1 B1	4	ft 1st B1 provided det $\neq 0$ ft 2nd B1 in non-trivial cases
(ii)	$\mathbf{M}=\frac{1}{2}\begin{bmatrix}1 & 1\\ 2 & 4\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & -3\end{bmatrix}\begin{bmatrix}4 & -1\\ -2 & 1\end{bmatrix}$	M1		Some attempt at mtx. multn.
	$=\frac{1}{2}\begin{bmatrix}1 & 1\\ 2 & 4\end{bmatrix}\begin{bmatrix}12 & -3\\ 6 & -3\end{bmatrix}$ or $\frac{1}{2}\begin{bmatrix}3 & -3\\ 6 & -12\end{bmatrix}\begin{bmatrix}4 & -1\\ -2 & 1\end{bmatrix}$	A1		First multn. correct ft
	$=\begin{bmatrix}9 & -3\\ 24 & -9\end{bmatrix}$	A1	3	Ft missing $\frac{1}{2}$ only
(b)(i)	When n even, $\mathbf{D}^n=\begin{bmatrix}3^n & 0\\ 0 & 3^n\end{bmatrix}$	M1		Incl. use in mtx. multn. of form $\mathbf{U}\,\mathbf{D}^n\,\mathbf{U}^{-1}$
	$\mathbf{M}^n=\frac{1}{2}\begin{bmatrix}1 & 1\\ 2 & 4\end{bmatrix}\begin{bmatrix}4.3^n & -3^n\\ -2.3^n & 3^n\end{bmatrix}$ or $\frac{1}{2}\begin{bmatrix}3^n & 3^n\\ 2.3^n & 4.3^n\end{bmatrix}\begin{bmatrix}4 & -1\\ -2 & 1\end{bmatrix}$ correct	A1		Correct ft
	Showing $\mathbf{M}^n=3^n\,\mathbf{I}$ legitimately	A1	3	
(ii)	When n odd, $\mathbf{D}^n=\begin{bmatrix}3^n & 0\\ 0 & -3^n\end{bmatrix}$	M1		Incl. use in mtx. multn. of form $\mathbf{U}\,\mathbf{D}^n\,\mathbf{U}^{-1}$
	$\mathbf{M}^n=\frac{1}{2}\begin{bmatrix}1 & 1\\ 2 & 4\end{bmatrix}\begin{bmatrix}4.3^n & -3^n\\ 2.3^n & -3^n\end{bmatrix}$ or $\frac{1}{2}\begin{bmatrix}3^n & -3^n\\ 2.3^n & -4.3^n\end{bmatrix}\begin{bmatrix}4 & -1\\ -2 & 1\end{bmatrix}$ correct	A1		Correct ft
	Showing $\mathbf{M}^n=3^{n-1}\,\mathbf{M}$ legitimately	A1	3	
	Total		**13**	

Question 8

8 (a) Matrix $\mathbf{M} = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$. Without attempting to factorise, expand fully $\det \mathbf{M}$.

(2 marks)

(b) Matrix $\mathbf{N} = \begin{bmatrix} d & e & f \\ f & d & e \\ e & f & d \end{bmatrix}$. Find the product matrix $\mathbf{MN}$. *(3 marks)*

(c) Prove that the product

$$(a^3 + b^3 + c^3 - 3abc)(d^3 + e^3 + f^3 - 3def)$$

can be written in the form $x^3 + y^3 + z^3 - 3xyz$, stating clearly each of x, y and z in terms of a, b, c, d, e and f. *(2 marks)*

Student Response

Question number | | Leave blank

8.

a) $a(a^2 - bc) - b(ca - b^2) + c(c^2 - ab)$ M1

$= a^3 - abc + b^3 - abc + c^3 - abc$

$= a^3 + b^3 + c^3 - 3abc$ A1 | 2

b) $\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} \times \begin{bmatrix} d & e & f \\ f & d & e \\ e & f & d \end{bmatrix}$

$= \begin{bmatrix} ad + bf + ce & ae + bd + cf & af + be + cd \\ cd + af + be & ce + ad + bf & cf + ae + bd \\ bd + cf + ae & be + cd + af & bf + ce + ad \end{bmatrix}$ M1, A2 | 3

c) let $x = ad + bf + ce$

$y = ae + bd + cf$

$z = af + be + cd$

$a^3 + b^3 + c^3 - 3abc = |M|$

$d^3 + e^3 + f^3 - 3def = |N|$

$MN = \begin{bmatrix} x & y & z \\ z & x & y \\ y & z & x \end{bmatrix}$ $\det = x^3 + y^3 + z^3 - 3xyz.$

blank

$$|MN| = |M||N|$$

$$(a^3+b^3+c^3-3abc)(d^3+e^3+f^3-3def)$$ M1

$$= x^3+y^3+z^3-3xyz$$

when $x = ad+bf+ce$

$y = ae+bd+cf$

$z = af+be+cd$. A1

2

7

Commentary

This candidate has found the determinant of $\mathbf{M}$, correctly multiplying out the brackets and collecting the like terms. The ambiguity about the power of b in the 2nd line was given the benefit of the doubt by the examiner as it was written correctly in the 3rd line. Candidates must remember, though, that if they alter a number or letter it should be rewritten clearly.
In part (b), a significant number of candidates tried to find the **determinant** of $\mathbf{MN}$ instead of multiplying the matrices and there were many errors made from mixing up e and c or b and d. No such problems here and the corrections are clear so full marks.
In part (c) this candidate spotted that the matrix found in (b) is of the same form as $\mathbf{M}$ and $\mathbf{N}$. That recognition, together with the result of (a), led to the correct identification of x, y and z. This alone would have gained one B mark as a special case. However, the candidate thought carefully and identified $(a^3+b^3+c^3-3abc)(d^3+e^3+f^3-3def)$ in the question as the product $\det\mathbf{M} \times \det\mathbf{N}$, so getting the left-hand side of the required result. The $\det(\mathbf{MN})$ was then shown to be the right-hand side of the required result. The candidate next wrote down, and then used, the formula $\det\mathbf{MN} = \det\mathbf{M} \times \det\mathbf{N}$ to complete the proof. Both marks have been gained even without the final summing up.

Mark Scheme

8(a)	$\text{Det}(\mathbf{M}) = a^3+b^3+c^3-3abc$	M1 A1	2	Good attempt; correct
(b)	$\begin{bmatrix} ad+bf+ce & ae+bd+cf & af+be+cd \\ af+be+cd & ad+bf+ce & ae+bd+cf \\ ae+bd+cf & af+be+cd & ad+bf+ce \end{bmatrix}$	M1 A1 A1	3	At least 5 correct; all 9 correct
(c)	Use of $\det(\mathbf{MN}) = \det(\mathbf{M})\det(\mathbf{N})$ $x = ad+bf+ce$, $y = ae+bd+cf$ and $z = af+be+cd$	M1 A1	 2	 All correctly identified Give B1 (SC) if just this with no explanation why
	Total		7	